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Convergence of Dynamical Zeta Functions

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I study poles and zeros of zeta functions in one-dimensional maps. Numerical and analytical arguments are given to show that the first pole of one such zeta function is given by the first zero of *another* zeta function: this describes convergence of the calculations of the first zero, which is generally the physically interesting quantity. Some remarks on how these results should generalize to zeta functions of dynamical systems with "pruned" symbolic dynamics and in higher dimensions follow.

KEY WORDS: Zeta functions; thermodynamic formalism in dynamical systems.

INTRODUCTION

A considerable part of the investigations of chaotic nonlinear systems has been devoted to the study of various characteristic numbers; Lyapunov exponents for the divergence of nearby trajectories, fractal dimensions for the intricate shapes of attractors and repellers, and entropies describing the loss of information of initial conditions over time.[1]

Other examples of characteristic numbers are escape rates[2] from repellers, resonances in correlation functions,[3] and semiclassical eigenvalues of the corresponding quantum problem for conservative systems.[4,5]

Averages like Lyapunov exponents are invariant under smooth deformations of the representation of a dynamical system, in particular in what set of coordinates it is displayed, but they are normally calculated by brute force. This is unsatisfying if one has the (very distant) aim of classifying chaotic dynamic behavior, and can also be bothersome for practical purposes if convergence to the asymptotic value is slow. One would therefore

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like to have a more analytic object to study, from which one could calculate the averages, and which would hopefully converge in a nice way.

Now, there is such a characteristic function, called, by formal analogy to Euler's product formula for Riemann's zeta function, an (inverse) dynamical zeta function[6]:

$$\zeta^{-1}(z, A)=\prod_{p}\left[1-z^{p} \exp \sum_{k=0}^{n-1} A(f^{k}(x))\right] \tag{1}$$

where the product goes over p, the primitive (nonrepeating) orbits of a map $x \rightarrow f(x)$; p also denotes the length of the orbit in numbers of iteration steps, and A is smooth function on state space. If prefer to write such a function of z as

$$\zeta^{-1}(z)=\prod_{p}(1-z^{p} t_{p})=\sum z^{k} c_{k} \tag{2}$$

where t_p, the quantity associated to periodic orbit p, could be negative in sign.

Although in principle straightforward, it is not widely known how to write down the appropriate zeta function for a corresponding average quantity, so for completeness I repeat the derivation of how to calculate generalized Lyapunov exponents,[7] defined by

$$\Lambda_{\mu}(\beta)=\lim_{n \rightarrow \infty} \Lambda_{\mu}^{n}(\beta); \qquad \Lambda_{\mu}^{n}(\beta)=\frac{1}{n} \log \sum_{(i)} p_{i}\left|\frac{df^{n}}{dx_{i}}\right|^{\beta} \tag{3}$$

where p_i is the weight given by the measure $d\mu$ to the ith element of a partition of state space that is made finer as n tends to infinity. The standard (leading) Lyapunov exponent is related to the generalized ones by

$$\lambda_{\mu}=\left.\frac{d\Lambda_{\mu}(\beta)}{d\beta}\right|_{\beta=0}=\frac{1}{n} \sum_{(i)} p_{i} \log \left|\frac{df^{n}}{dx_{i}}\right| \tag{4}$$

At least for hyperbolic systems, one can choose a partition after periodic orbits,[8] such that, for the natural measure on chaotic attractors with one expanding direction, the sum runs over periodic orbits of length n, and $p_i \sim |D_i|^{-1}=|df^n/dx_i|^{-1}$. If there are many expanding directions, one should take the inverse of $|L_i|$, the absolute value of the product of all expanding eigenvalues of the derivative matrix, and on a repeller one would have a normalize the total measure to one by the escape rate, which can be found as the exponential decrease of $\sum_{(i)} |L_i|^{-1}$.[2] In any of these cases, p_i is a calculable quantity that only depends on the eigenvalues derivative matrices of periodic orbits, which are invariant quantities.

Hence, in the sense of logarithms,

$$e^{n\Lambda_\mu(\beta)} \sim Z_n(\beta) = \sum_{(i)} p_i \left| \frac{df^n}{dx_i} \right|^\beta \tag{5}$$

and we can also calculate $\Lambda(\beta)$ from the value of z at which the generating function

$$\Omega(z, \beta) = \sum z^n Z_n(\beta) \tag{6}$$

diverges. For a one-dimensional attractor, Ω is connected to the zeta function

$$\zeta^{-1}(z, \beta - 1) = \prod_p (1 - z^p |D_p|^\beta |D_p|^{-1}) \tag{7}$$

through $\Omega(z, \beta) = -(zd/dz) \log \zeta^{-1}(z, \beta - 1)$. With more expanding directions and on repellers, one would have to modify the term $|D_p|^{-1}$ as described above. $\Lambda(\beta)$ is hence $-\log z_0$, where z_0 is the first positive zero of $\zeta^{-1}(z, \beta - 1)$.

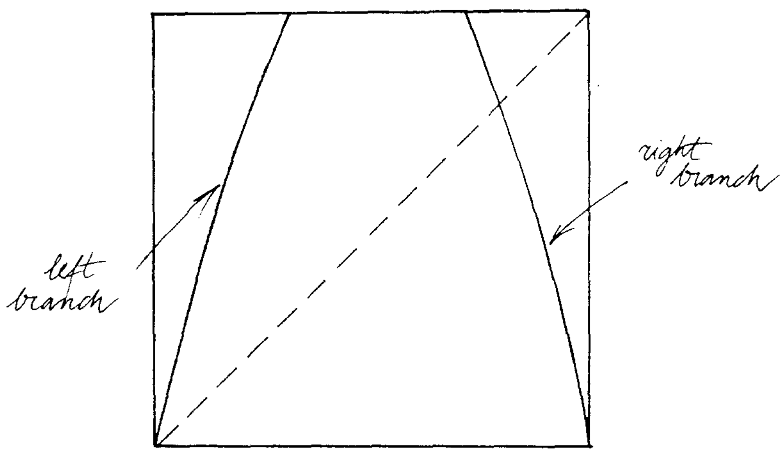
Only recently has it become clear how to turn this formula into an efficient means of evaluating the average quantity. The simple recipe is to treat the inverse zeta function as an ordinary function in the complex variable z, and calculate the location of the first zero by truncating the power series to a finite polynomial.[9] For this to work, the convergence radius has to stretch at least to the location of the first zero: in the ideal case convergence is limited by a pole at z_1 of larger absolute value; then $c_k \sim z_1^{-k}$, and around z_0 the terms in the power series go down exponentially as $(z_0/z_1)^k$. The convergence of the calculation of z_0 is hence exponential with the cutoff, and it suffices to know relatively few short periodic orbits to arrive at a good estimate of the correct answer. Rigorous results of Ruelle and others[6,10] show that for hyperbolic systems, and the quantity evaluated being multiplicative over the points of the orbit as in Eq. (1), the smallest zero of the inverse zeta function is indeed isolated, and the power series converges beyond that first zero.

My aim in this paper is to calculate z_1, that is, just how far the power series converges. More modestly, my numerical examples are from one-dimensional expanding (but nonlinear) maps, and the arguments are phrased for that case. In the discussion section I try to argue that though similar results are expected to hold for hyperbolic systems, when the the quantity associated with an orbit is multiplicative over the points of the orbit, they should fail if any of these conditions are not fulfilled. The latter condition excludes, for instance, Lyapunov exponents if there are several expanding ones, but not the escape rate from a repeller, which is determined by the product of the expanding eigenvalues.

1. COMBINATORICS OF THE ZETA FUNCTION

The underlying reason why the power series of ζ^{-1} converges faster than one would expect judging from the size of the factors in the representation as an infinite product (in other words: why the first pole is farther from the origin that the first zero) is that the coefficients of the higher orders in z can be written as offsets between primitive orbits, and estimates of the same from shorter orbits. For these offsets Cvitanović has coined the term "curvatures,"[9] and when the estimates are close, as they are in hyperbolic systems, the curvatures will be small.

For definitiveness, consider a one-dimensional repeller: the non-wandering points of an overshooting tent map (see Fig. 1). The orbits are then labeled by sequences in the symbols 0 and 1, corresponding to passages on the left-hand and right-hand sides of the "hole." The primitive orbits are labeled by periodic sequences in the symbols 0 and 1 that are



(a)

Fig. 1. (a) An overshooting tent map. The symbolic itinerary of a point is given by "0" on each passage on the left-hand side of the hole, and "1" on each passage on the right-hand side. Infinite symbol sequences correspond to points that stay in the interval for all times. (b) A two-scale approximation to the map in panel (a). (c) A four-scale approximation to the map in panel (a).

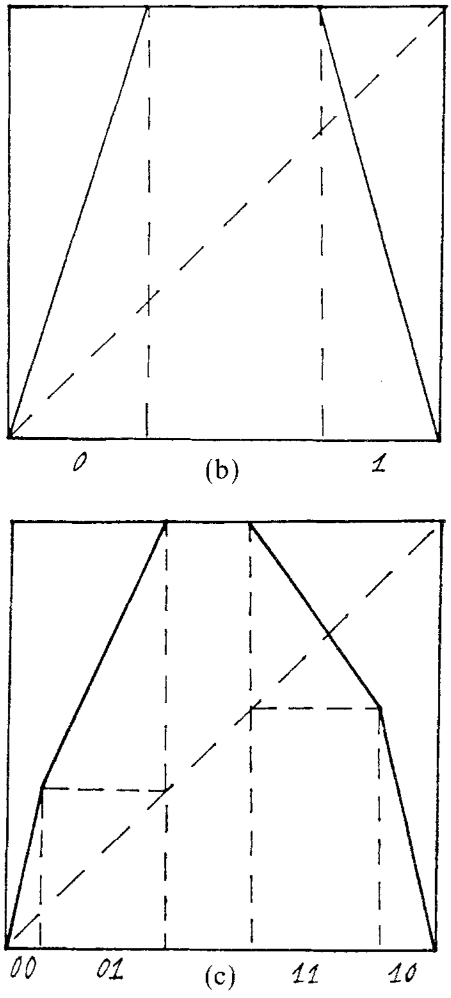


Fig. 1. (*Continued*)

not repeats of shorter periodic sequences and counted modulo circular permutation. The zeta function written out then looks like

$$\begin{aligned}\zeta^{-1}(z) = 1 &- z(t_0 + t_1) - z^2(t_{01} - t_0 \cdot t_1)\\ &- z^3(t_{001} - t_0 \cdot t_{01} + t_{011} - t_{01} \cdot t_1)\\ &- z^4(t_{0001} - t_0 \cdot t_{001} + t_{0011} - t_0 \cdot t_{011}\\ &- t_{001} \cdot t_1 + t_0 \cdot t_1 \cdot t_{01} + t_{0111} - t_{011} \cdot t_1)\\ &- z^5(\cdots)\end{aligned} \tag{8}$$

Up to this order the choice of "counterterms" to a given periodic orbit is compelling: it is clearly the terms of type $t_{01} - t_0 \cdot t_1$, $t_{001} - t_0 \cdot t_{01}$,..., that we would like to identify as curvatures. It seems hard, though, to give a combinatorial definition that works on all orbits to all orders in z, so in the following I address the more limited goal of rearranging the periodic orbits in terms whose offsets are small. Probably they are not optimal choices for curvatures, in so far as such a choice exists, but nevertheless one can draw some useful conclusions about the convergence of the power series for ζ^{-1}.

Suppose now that all quantities t_s only depend on finitely many bits in the binary address; this amounts to assuming that the map is piecewise linear, and that the different linear pieces map onto each other appropriately (see Fig. 1b and 1c). Feigenbaum[11] has shown how to describe this motion by Markov diagrams that code the passage between the different subintervals (see Fig. 2). This entails that a cycle derivative, say D_{00101} is approximated by

$$\begin{aligned} &\sigma_0\sigma_0\sigma_1\sigma_0\sigma_1 \quad (N=0)\\ &\sigma_{00}\sigma_{01}\sigma_{10}\sigma_{01}\sigma_{10} \quad (N=1)\\ &\sigma_{001}\sigma_{010}\sigma_{101}\sigma_{010}\sigma_{100} \quad (N=2)\\ &\sigma_{0010}\sigma_{0101}\sigma_{1010}\sigma_{0100}\sigma_{1001} \quad (N=3) \end{aligned} \tag{9}$$

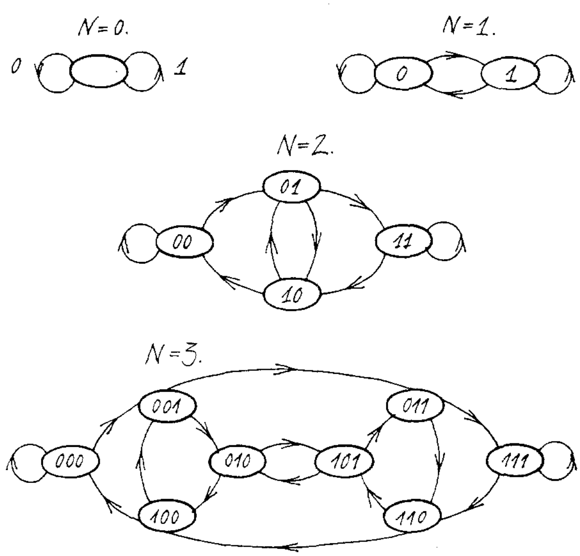


Fig. 2. The Markov diagrams describing better and better approximations to the motion under iterations of the tent map.

where the σ's are factors attached to the links in the Markov diagrams, and can be identified as slopes in the corresponding intervals of the piecewise linear map. The zeta function approximating all quantities to have finite dependence on the coding symbols is

$$\prod_p (1 - z^p t_p) = \sum z^k a_k \tag{10}$$

where in the Nth approximation, a_k is the sum over all closed paths of length k that do not cut themselves on the Nth graph and all products of shorter closed paths with combined length k that do not cut themselves and do not mutually intersect, each path in each product with minus sign. Let us consider one closed path that cuts itself on the $N = 3$ graph, (00101) (see Fig. 3).

The crossing-node is marked 010. In the full zeta function there is a term $z^5(t_{001} \cdot t_{01})$, which on this graph is not distinguishable from $z^5 t_{00101}$, and therefore cancels. The combination

$$\langle 00101 \rangle = 00101 - 001 * 01 \equiv t_{00101} - t_{001} \cdot t_{01} \tag{11}$$

should be generally be small.

Let us formalize the procedure: start with any periodic sequence in the symbols. In a sufficiently high-N graph the corresponding loop does not cut itself, but if we descend down the hierarchy of approximations we will find one where it does so, which defines what we may call the *order* of a cycle. As a mnemonic device we call the collection of terms in the full product that fit the cycle on the graph of its order a *complex*. A term in a complex, a cycle or product of cycles, can be referred to as a *radical*. A list of complexes with their radicals by length in a binarily coded set is given in Table I.

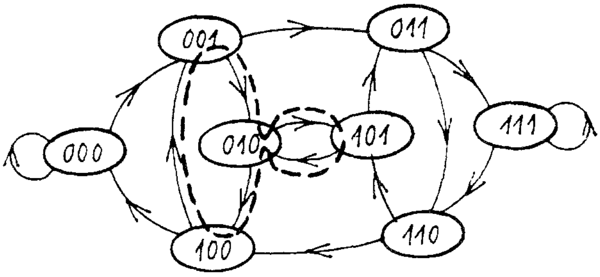


Fig. 3. The path of the cycle 00101 traced on the $N = 3$ Markov diagram.

Table I

Of length = 2 complexes are

$$\langle 01\rangle = 01 - 0*1$$

Of length = 3 complexes are

$$\langle 001\rangle = 001 - 01*0$$
$$\langle 011\rangle = 011 - 01*1$$

Of length = 4 complexes are

$$\langle 0001\rangle = 0001 - 001*0$$
$$\langle 0011\rangle = 0011 - 001*1 - 011*0 + 01*0*1$$
$$\langle 0111\rangle = 0111 - 011*1$$

Of length = 5 complexes are

$$\langle 00001\rangle = 00001 - 0001*0$$
$$\langle 00011\rangle = 00011 - 0011*0$$
$$\langle 00101\rangle = 00101 - 001*01$$
$$\langle 00111\rangle = 00111 - 0011*1$$
$$\langle 01011\rangle = 01011 - 01*101$$
$$\langle 01111\rangle = 01111 - 0111*1$$

Of length = 6 complexes are

$$\langle 000001\rangle = 000001 - 00001*0$$
$$\langle 000011\rangle = 000011 - 00011*0$$
$$\langle 000101\rangle = 000101 - 0001*01$$
$$\langle 000111\rangle = 000111 - 00011*1 - 00111*0 + 0011*0*1$$
$$\langle 001101\rangle = 001101 - 0011*10 - 001*011 + 001011$$
$$\langle 001111\rangle = 001111 - 00111*1$$
$$\langle 010111\rangle = 010111 - 01*1011$$
$$\langle 011111\rangle = 011111 - 01111*1$$

Of length = 7 complexes are

$$\langle 0000001\rangle = 0000001 - 000001*0$$
$$\langle 0000011\rangle = 0000011 - 000011*0$$
$$\langle 0000101\rangle = 0000101 - 00001*01 - 000101*0 + 0001*0*01$$
$$\langle 0000111\rangle = 0000111 - 000111*0$$
$$\langle 0001001\rangle = 0001001 - 0001*001$$
$$\langle 0001101\rangle = 0001101 - 00011*10 - 0001*011 + 0001011$$
$$- 001101*0 + 0011*0*10 + 001*0*011 - 001011*0$$
$$\langle 0001111\rangle = 0001111 - 000111*1$$
$$\langle 0010011\rangle = 0010011 - 001*1001$$
$$\langle 0010101\rangle = 0010101 - 00101*01$$
$$\langle 0011011\rangle = 0011011 - 0011*011$$
$$\langle 0011101\rangle = 0011101 - 00111*10 - 001101*1 + 0011*1*10$$
$$- 001*0111 + 0010111 + 001*011*1 - 001011*1$$
$$\langle 0011111\rangle = 0011111 - 001111*1$$
$$\langle 0101011\rangle = 0101011 - 01*10101$$
$$\langle 0101111\rangle = 0101111 - 010111*1 - 01*10111 + 01*1011*1$$
$$\langle 0110111\rangle = 0110111 - 011*1101$$
$$\langle 0111111\rangle = 0111111 - 011111*1$$

Table continued

Table I (*continued*)

Of length = 8 complexes are

$\langle 00000001 \rangle = 00000001 - 0000001 * 0$
$\langle 00000011 \rangle = 00000011 - 0000011 * 0$
$\langle 00000101 \rangle = 00000101 - 0000101 * 0$
$\langle 00000111 \rangle = 00000111 - 0000111 * 0$
$\langle 00001001 \rangle = 00001001 - 00001 * 001$
$\langle 00001011 \rangle = 00001011 - 0001011 * 0$
$\langle 00001101 \rangle = 00001101 - 0001101 * 0$
$\langle 00001111 \rangle = 00001111 - 0000111 * 1 - 0001111 * 0 + 000111 * 0 * 1$
$\langle 00010101 \rangle = 00010101 - 000101 * 01$
$\langle 00011001 \rangle = 00011001 - 00011 * 100 - 0001 * 0011 + 00010011$
$\langle 00011011 \rangle = 00011011 - 00011 * 011$
$\langle 00011101 \rangle = 00011101 - 000111 * 10 - 0001101 * 1 + 00011 * 1 * 10$
$\quad - 0001 * 0111 + 00010111 + 0001 * 011 * 1 - 0001011 * 1$
$\quad - 0011101 * 0 + 00111 * 0 * 10 + 001101 * 0 * 1 - 0011 * 0 * 1 * 10$
$\quad + 001 * 0 * 0111 - 0010111 * 0 - 001 * 0 * 011 * 1 + 001011 * 0 * 1$
$\langle 00011111 \rangle = 00011111 - 0001111 * 1$
$\langle 00100101 \rangle = 00100101 - 001 * 01001$
$\langle 00100111 \rangle = 00100111 - 001 * 10011$
$\langle 00101011 \rangle = 00101011 - 001011 * 01$
$\langle 00101101 \rangle = 00101101 - 00101 * 101 - 001 * 01011 + 001 * 01 * 101$
$\langle 00101111 \rangle = 00101111 - 0010111 * 1$
$\langle 00110101 \rangle = 00110101 - 001101 * 10$
$\langle 00111011 \rangle = 00111011 - 00111 * 110 - 0011 * 0111 + 00110111$
$\langle 00111101 \rangle = 00111101 - 0011101 * 1$
$\langle 00111111 \rangle = 00111111 - 0011111 * 1$
$\langle 01010111 \rangle = 01010111 - 01 * 101011$
$\langle 01011011 \rangle = 01011011 - 01011 * 101$
$\langle 01011111 \rangle = 01011111 - 0101111 * 1$
$\langle 01101111 \rangle = 01101111 - 011 * 11011$
$\langle 01111111 \rangle = 01111111 - 0111111 * 1$

Unfortunately, complexes do not exhaust the terms produced by the zeta function, and are therefore not immediately good candidates for curvatures. Consider a collection of loops, which we symbolize by a collective index γ. At a sufficiently high-N graph all the loops are free and do not intersect. Descending in the hierarchy, one finds a diagram where either a loop intersects itself or several loops intersect. Only if all the loops connect at once are they represented by a radical in a complex, otherwise one may graphically "fill up" the intersecting loops and write

$$\prod_{p \in \gamma} z^p t_p \subset \left(\prod_{\substack{\text{complexes at} \\ \text{order } N}} z^c t_c \right) \left(\prod_{\substack{\text{loops free} \\ \text{at order } N}} z^p t_p \right) \tag{12}$$

The process is redone for the remaining loops:

$$\prod_{p \in \gamma} z^p t_p \subset \left(\prod_{\substack{\text{complexes} \\ \text{not connected}}} z^c t_c \right) \tag{13}$$

The restriction of nonconnectedness means that no two complexes may have loops in radicals in common or jointly be parts of another complex. The zeta function may thus be written as a sum over (restricted) products of complexes:

$$\prod (1 - z^p t_p) = \sum_{\substack{(c_1, \ldots, c_l) \\ \text{not connected}}} (-)^l \left(\prod_{i=1}^{l} z^{c_i} t_{c_i} \right) \tag{14}$$

Alternatively, we can continue to shrink the complexes below their order: when the graphs cut themselves in a new way we have further "secondary" radicals, and we can take care of the terms from the products over disconnected complexes this way. A possible definition of curvatures is then the collection of counterterms from primary and secondary radicals, the latter if necessarily partitioned between different complexes that coincide on lower order diagrams. As the secondary radicals can lie far from the primary in coordinate space, it is not clear if this is sensible, and I will not persue that question here.

2. ESTIMATES OF HYPERBOLIC COMPLEXES

In this section the close cancellations between primitive orbits and their "counterterms" in hyperbolic systems are used to find the first pole of ζ^{-1}. Let $t_p = |D_p|^\tau$ for some τ, and write the average of $\prod_{p \in \gamma} |D_p|^\tau$ as $t_c(\tau)$, when γ are the radicals of c. The radicals can always be joined in pairs of opposite sign that only differ in their choice of routing through one of the crossing-nodes of the graph of the complex. That node also in general indicates the region in coordinate space where the radicals are furthest apart, and where most of the difference of their derivative is generated. It is convenient not to consider directly the difference $D_\gamma^\tau - D_{\gamma'}^\tau$, but the the logarithm of the quotient between them:

$$\log \frac{D_\gamma}{D_{\gamma'}} = \sum_{\text{points along radical } \gamma} \int_{x_{\gamma'}}^{x_\gamma} N(f(x))\, dx$$

$$\sim \sum_{\text{points along radical } \gamma} N(f(x_\gamma)) \cdot (x_\gamma - x_{\gamma'}) \tag{15}$$

where N, the nonlinearity, is

$$N(f(x)) = \frac{d \log f'(x)}{dx} = \frac{f''(x)}{f'(x)} \tag{16}$$

If we have a bound N on the nonlinearity, we may approximate $D_{\gamma'}^{\tau}$ by $D_{\gamma}^{\tau} \exp(\tau N I)$, where I is the offset between the radicals in coordinate space where they are furthest apart. That offset is well approximated by the inverse derivative of the cycle corresponding to the symbolic address of the node where their choice of routing is different, and can be estimated to be something like λ^{-O} in size, if λ is an average expansion rate such as Lyapunov exponent, and O is the order. Therefore the absolute value of the difference is roughly

$$|D_{\gamma}^{\tau} - D_{\gamma'}^{\tau}| \sim t_c \cdot \lambda^{-O} \tau N \tag{17}$$

These approximations are valid if $\tau N \cdot |I|$ is small, or equivalently, for given τ when the order of the complex is high enough. One would like here to understand the distribution in orders of complexes of given length (large enough). This I have not done, more than to the extent that one can easily find the best and the worst cases. The best case is illustrated by complexes of the type

$$\langle 0 \cdots 01 \rangle = 0 \cdots 01 - 0 \cdots 1 * 0 \tag{18}$$

If their length is L, their order is $L-1$, and the offset kills one power of the derivative. Hence such curvatures are of order unity for all L, when τ is one.

On the other side of the spectrum we have two terms that correspond to two different tilings of an Euler path on a given Markov graph. An Euler path goes through every link in a graph precisely once. If we consider the graphs of Fig. 2 at level n, they have 2^n nodes and 2^{n+1} links. An "Euler complex" has 2^n crossing-points, which implies that there are in all 2^{2^n} radicals, to compare with the total number of terms in the zeta function of that length: $2^{2^{n+1}-1}$. Thus, the order is $\log_2 L - 1$, and the naive estimate shows that a pair of radicals is not much smaller than the bare terms. There is, however, no reason not to believe that the signs of all pairs making up the big complex fluctuate more or less randomly, so the argument does not necessarily imply slow convergence. In any case, it is clear by counting that the "worst" cases cannot amount to more than a vanishingly small fraction of all terms. A reasonable guess of the distribution of orders of complexes with given length would be that the typical order grows as some coefficient of the length between 0 and 1. If so, it follows that the sum over all complexes and products over disconnected complexes, the term multiplying z^k in the zeta function, converges exponentially fast with k.

However, by a heuristic argument we can do better and actually compute the convergence rate of the c_k, which will turn out to behave essentially as the best case. The argument parallels Feigenbaum's original derivation of the zeta function (in a finite approximation) determining the point of divergence of a grand partition function[11]: we will treat the zeta function summed order to order in z as another grand partition function and find the zeta function determining *its* divergence. Any pair of radicals looks essentially like the picture in Fig. 4, where the loops are embedded in the graph of the order of the complex, and if necessary other loops multiplying both terms. Now sum up all pairs of radicals corresponding to multiple traversals of a:

$$(ab - a * b) + (a^2b - a * ab) + (a^3b - a * a^2b) + \cdots \tag{19}$$

The pair of type $(a^Nb - a * a^{N-1}b)$ will be roughly $z^{|b|}\,|D_b|^\tau \cdot z^{|a|N} \cdot |D_a|^{\tau N}\,(D_a)^{-N}$, where one should keep in mind that the sign of D_a remains in the second term, as it comes from the offset in coordinate space. The tail sums up to

$$\frac{1}{1 - z^{|a|} \cdot |D_a|^\tau\,(D_a)^{-1}} \tag{20}$$

We thus arrive at the prediction that

$$\zeta^{-1}(z, \tau) = \prod_p (1 - z^p\,|D_p|^\tau) \sim \sum_p \frac{\cdots}{1 - z^{|a|} \cdot |D_a|^\tau\,(D_a)^{-1}} \tag{21}$$

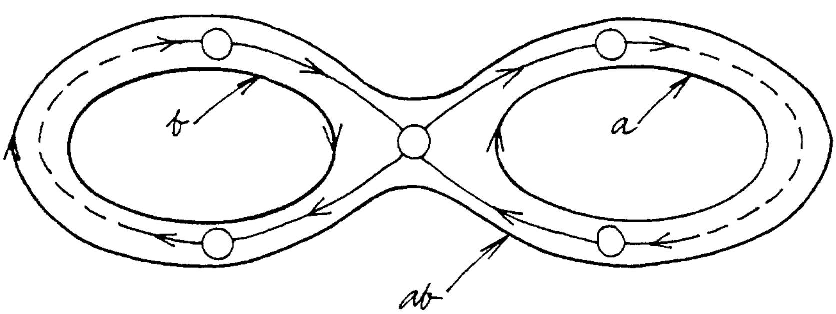


Fig. 4. One pair of radicals in a complex that only differ in the choice of routing through one node.

diverges at the value of z where

$$\zeta_{(-)}^{-1}(z, \tau-1)=\prod_{p}\left[1-z^{p}\left|D_{p}\right|^{\tau}\left(D_{p}\right)^{-1}\right]=0 \tag{22}$$

a rather nice feature if true.

Figures 5a–5l present numerical results of a tent map, the representation function of the Feigenbaum scaling function.[12] This is is repeller, and the zeta functions $\zeta^{-1}(z, \tau)$ describe Lyapunov exponents of the motion remaining on the Cantor set for all time with all symbolic labels having equal measure, or fractal dimensions of that set with the same measure. The natural measure can be used in a similar way, as described in the introduction.[8] I have also investigated a fractional linear approximation of the representation function, with qualitatively the same results. The first three poles and zeros are extracted by constructing the [3/3] Padé approximation of the series (1). One may note that the smallest zero of the zeta function (22) is negative for small τ, but is overtaken by a positive zero at τ around 1/2, and that the same thing happens for leading pole of the zeta function (21). Except for cusps around $\tau=0$ and other points caused by cancellation errors when poles and nonleading zeros of the same zeta function collide, evidently not only the first pole of (21), but also the next is very well matched by zeros of (22). Conversely, the poles of (22) are well matched by the zeros of (21), with τ decreased by two.

In the next subsection I argue that a probable reason for this is that the infinite product of (inverse) zeta functions

$$Z(z, \tau)=\prod_{l=0}^{\infty}\left\{\prod_{p}\left[1-z^{p}\left|D_{p}\right|^{\tau}\left(D_{p}\right)^{-l}\right]\right\} \tag{23}$$

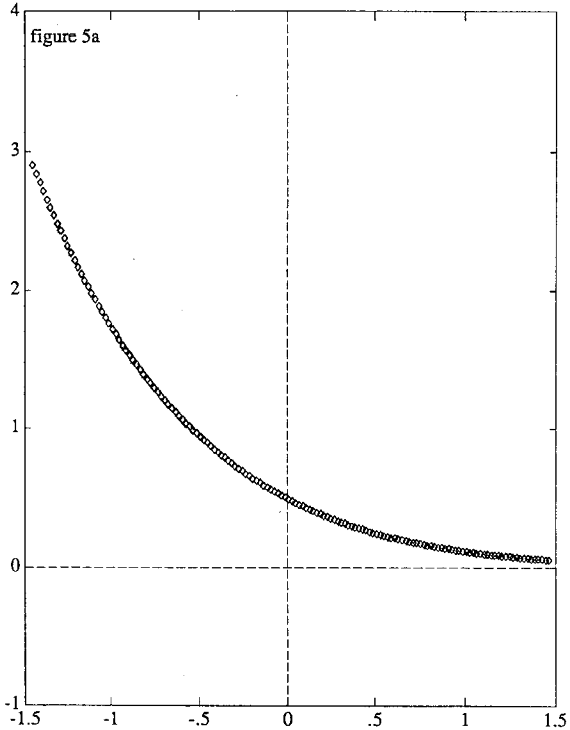
is in fact an entire analytic function of z. If so,

$$\zeta^{-1}(z, \tau)=\frac{Z(z, \tau)}{Z_{(-)}(z, \tau-1)} \tag{24}$$

where

$$\begin{aligned} Z_{(-)}(z, \tau-1) &=\prod_{l=0}^{\infty}\left\{\prod_{p}\left[1-z^{p}\left|D_{p}\right|^{\tau}\left(D_{p}\right)^{-1}\left(D_{p}\right)^{-l}\right]\right\} \\ &=\zeta_{(-)}^{-1}(z, \tau-1)\, Z(z, \tau-2) \end{aligned} \tag{25}$$

and so at least the first poles of $\zeta^{-1}(z, \tau)$ coincide with the zeros of $\zeta_{(-)}^{-1}(z, \tau-1)$.



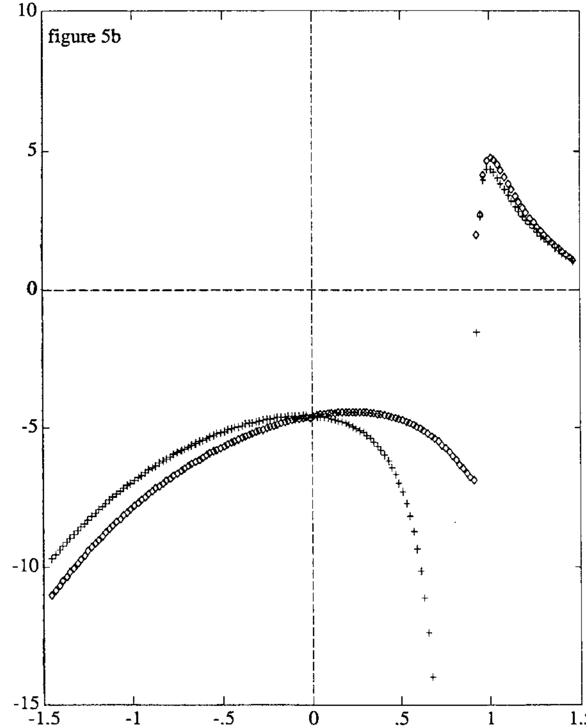


Fig. 5. (a–j) The calculated values of poles and zeros around $\tau = 0$: the different branches are continued when they change order in absolute size. (k, l) $q = -\log_2 |z|$ sorted in absolute size over a wider region. (a) The first (physical) zero of the (inverse) zeta function (16). (b) The lower branch pole and lower branch zero of (16). Note that they collide at $\tau = 0$. (c) The lower branch pole of (16) and the first zero of (17). (d) The difference between the pole and zero in panel (c). (e) The second pole of (16) and the second zero of (17). (f) The difference between the pole and zero in panel (e). (g) The first pole of (17), and the first zero of (16), with τ shifted by 2. (h) The difference between the pole and zero in panel (g). (i) The second pole of (17), and the second zero of (16), with τ shifted by 2. (j) The difference between the pole and zero in panel (i). (k) The base-two logarithm of the first zero and smallest (in absolute value) pole of (16) with the smallest (in absolute value) zero of (17). (l) The difference between the logarithms of the pole from (16) and zero from (17) in panel (k).

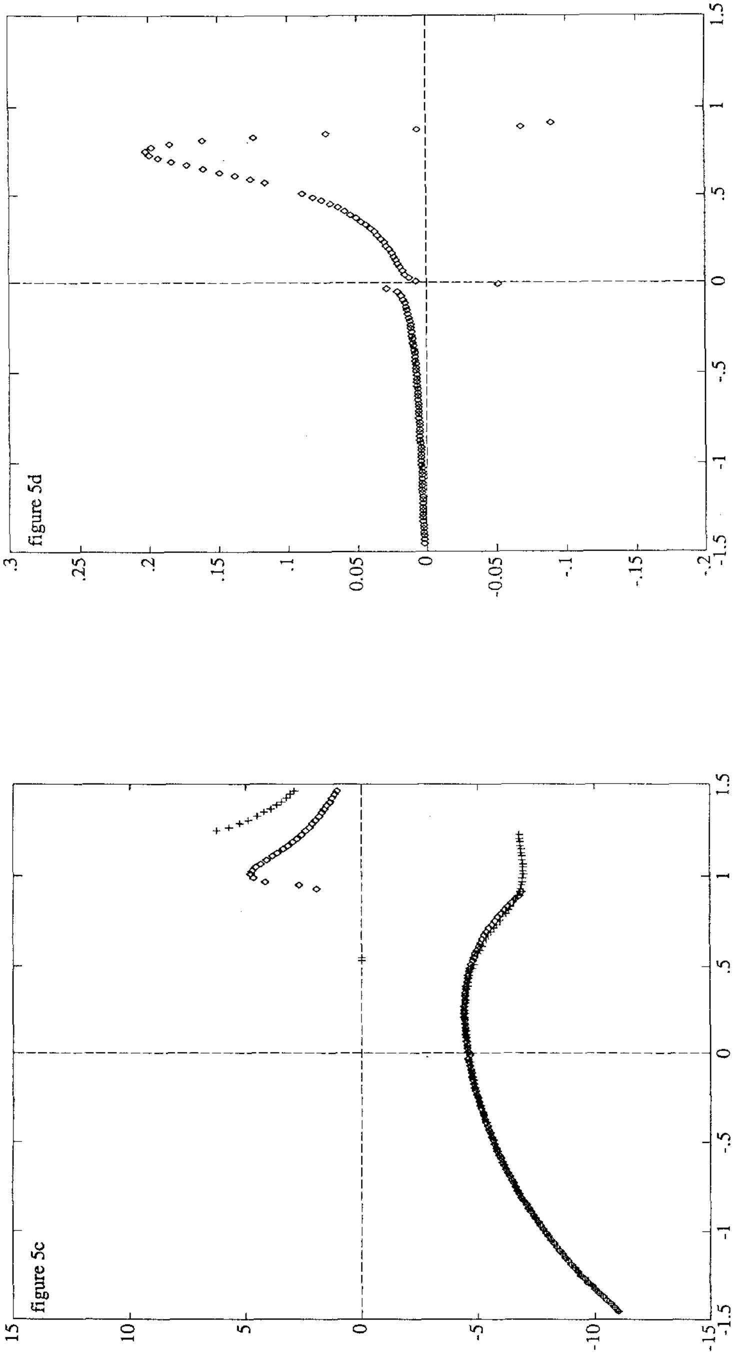


Fig. 5. *(Continued)*

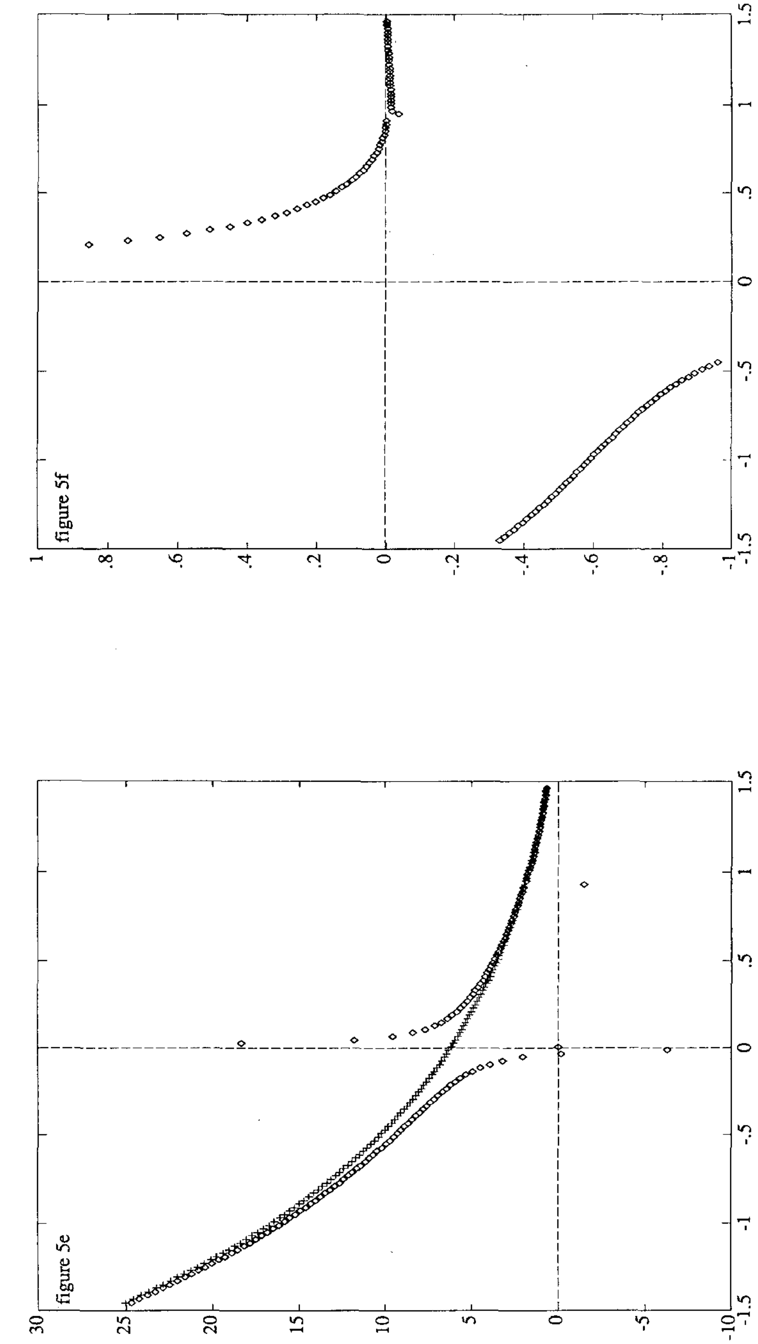


Fig. 5. *(Continued)*

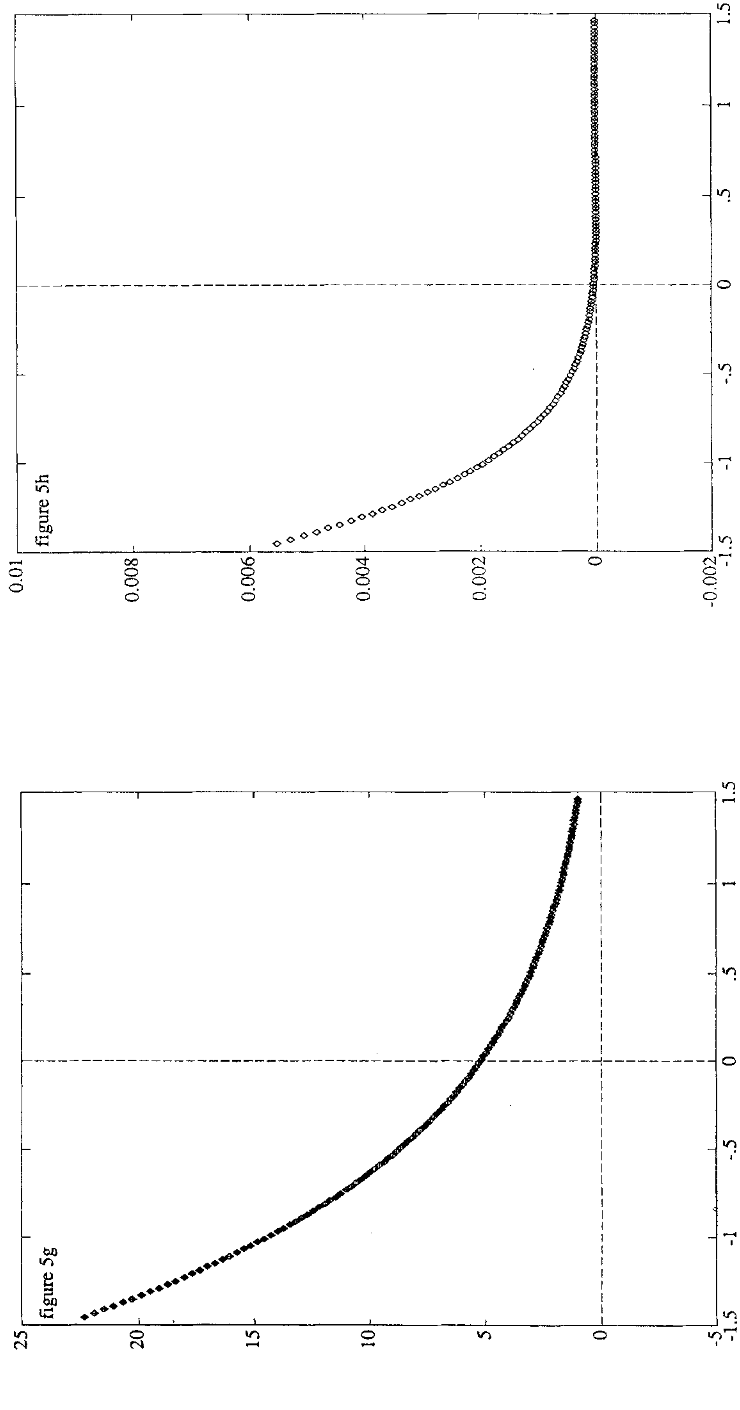
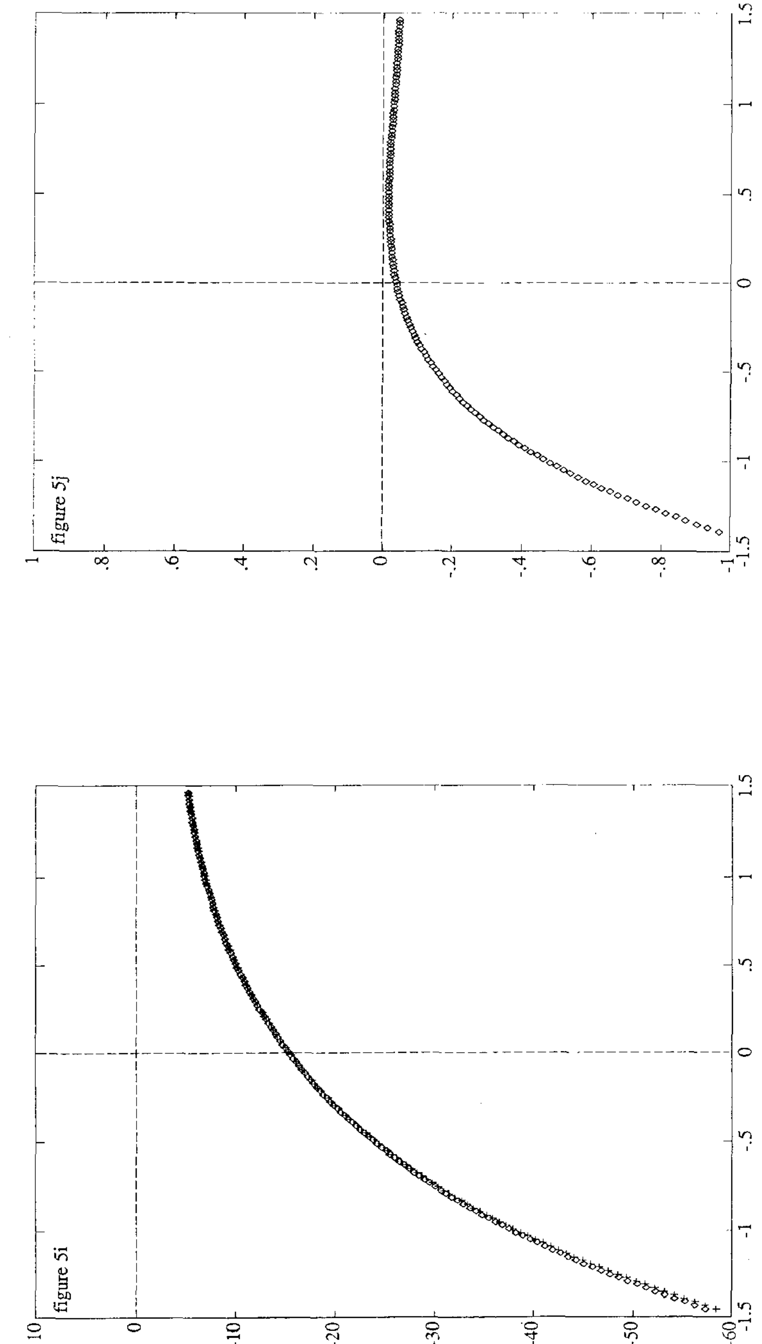


Fig. 5. *(Continued)*



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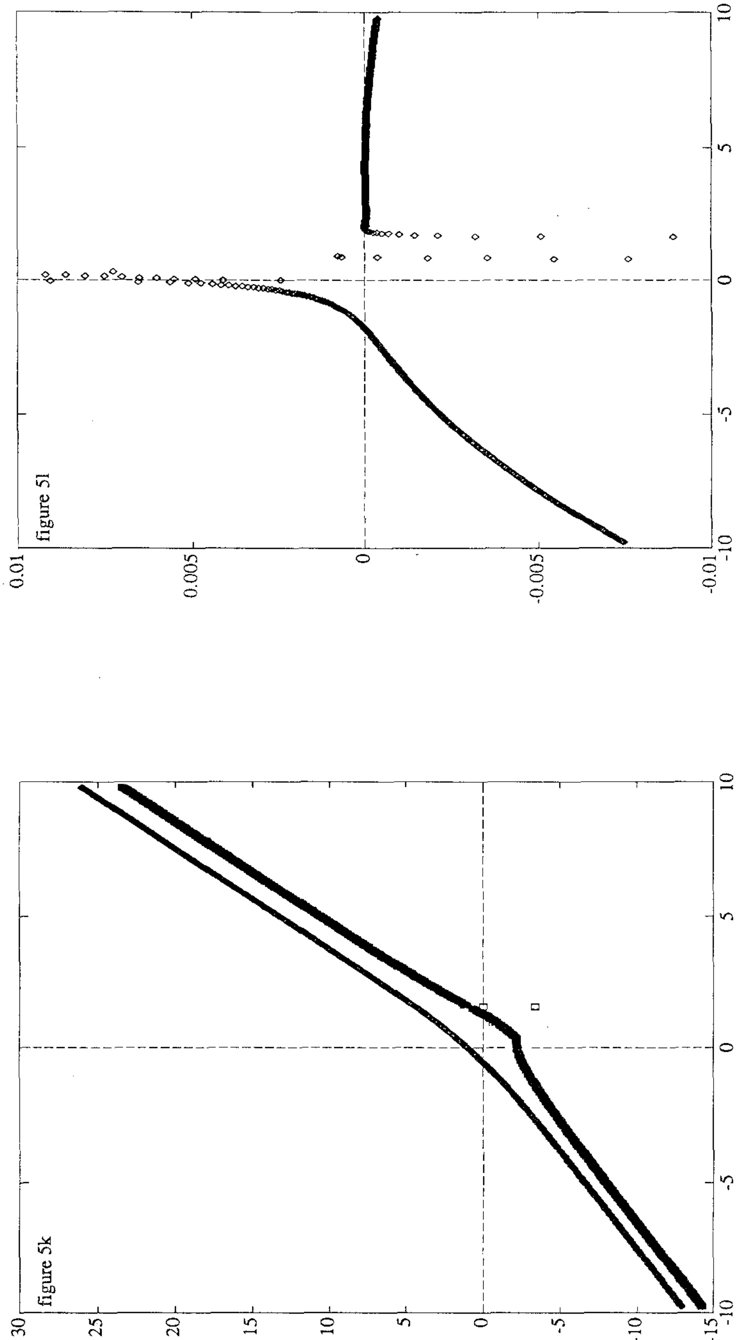

Fig. 5. (*Continued*)

3. ZETA FUNCTIONS AND INVERSE DETERMINANTS

This section contains two further arguments to determine the location of the first pole of ζ^{-1}, following the hypothesis at the end of the last section. Let us first reconsider the partition sum of Ruelle[6]:

$$Z_n(A) = \sum_{x \in \mathrm{Fix}\, f^n} \exp \sum_{k=0}^{n-1} A(f^k(x)) \tag{26}$$

where A is a continuous real function. As we know from the introduction, the exponential in n growth of Z_n is determined by the smallest zero of the appropriate zeta function:

$$\begin{aligned}
\sum_{n=1}^{\infty} z^n Z_n(A) &= \sum_p p(z^{|p|} \exp \sum_{k=0}^{n-1} A(f^k(x_p)) \\
&\quad + z^{2|p|} \exp 2 \sum_{k=0}^{n-1} A(f^k(x_p)) + \cdots) \\
&= -z \frac{d}{dz} \log \prod_p \left[1 - z^p \exp \sum_{k=0}^{n-1} A(f^k(x)) \right] \\
&\equiv -z \frac{d}{dz} \log \zeta^{-1}(z; A)
\end{aligned} \tag{27}$$

If all the periodic orbits are labeled by symbolic addresses, the sum over periodic orbits can be identified as a sum over symbol sequences with periodic boundary conditions. These sums can then also be considered as the traces of the nth powers of the kernel of the Ruelle–Araki operator, which acts on a function g defined on binary addresses as

$$(Lg)(i_1, i_2, \ldots) = \sum_{i_0} e^{A(i_0, i_1, i_2, \ldots)} g(i_0, i_1, i_2, \ldots) \tag{28}$$

If the operator has discrete eigenvalues λ_i, formally

$$\sum_{n=1}^{\infty} z^n Z_n(A) \sim \sum_i \frac{z\lambda_i}{(1 - z\lambda_i)}$$

so they are *a priori* related to zeros of ζ^{-1}. That the largest eigenvalue of L is isolated implies the result for the smallest zero of ζ^{-1} stated in the introduction: they are naturally related by $z_0 \lambda_0 = 1$. However, the numerical and analytic results of the preceding section show that ζ^{-1} has a pole for finite z, so it is clear that the relation does not extend to all the higher eigenvalues.

Following Kadanoff and Tang,[2] it would seem more natural to consider an operator acting as

$$(H\phi)(x) = \int \sum_{y=f^{-1}(x)} \delta(z-y)\, e^{A(z)} \phi(z)\, dz \tag{29}$$

The kernel of this integral operator is not bounded, but suppose for a moment that it is, for instance, by broadening the delta function to $\delta_\sigma(z-y)$. Then it would have a discrete spectrum, with eigenfunctions ϕ_n satisfying

$$(H_\sigma \phi_n)(x) = \lambda_n \int \sum_{y=f^{-1}(x)} \delta_\sigma(z-y)\, e^{A(z)} \phi_n(z)\, dz \tag{30}$$

and the eigenvalues real if the kernel is Hermitian, but otherwise complex. The proof of this is of course completely classical, but as the limit when the delta function becomes sharp again sheds light on the zeta functions, it is interesting to summarize a few results from Fredholm's theory.[13,14]

One can form the resolvent function G that satisfies a twinning relation with the kernel

$$H(x, z) = \sum_{y=f^{-1}(x)} \delta_\sigma(z-y) e^{A(z)}$$

as follows:

$$G(x, y, \lambda) = H(x, y) + \lambda \int H(x, s)\, G(s, y, \lambda)\, ds \tag{31}$$

and has the power series expansion

$$G(x, y, \lambda) = \sum_{n=0}^{\infty} \lambda^n \int ds_1\, ds_2 \cdots ds_n\, H(x, s_1)\, H(s_1, s_2) \cdots G(s_n, y) \tag{32}$$

convergent for small enough λ. One can write G as the quotient of two entire analytic functions $D(x, y, \lambda)$ and $D(\lambda)$ defined by Fredholm series as the sum of antisymmetrized traces of the kernel H:

$$\begin{aligned} D(\lambda) = 1 - \lambda \int H(x, x)\, dx & \\ + \frac{(-\lambda)^2}{2!} \iint \begin{vmatrix} H(x,x) & H(x,y) \\ H(y,x) & H(y,y) \end{vmatrix} dx\, dy & \\ + \frac{(-\lambda)^3}{3!} \iiint \begin{vmatrix} H(x,x) & H(x,y) & H(x,z) \\ H(y,x) & H(y,y) & H(y,z) \\ H(z,x) & H(z,y) & H(z,z) \end{vmatrix} dx\, dy\, dz & \\ + \cdots & \end{aligned} \tag{33}$$

and similarly for $D(x, y, \lambda)$. The D is connected to the trace of G in a very similar way as the zeta function is connected to the generating function:

$$-\frac{d}{d\lambda} D(\lambda) = \int ds\, G(s, s, \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} \operatorname{tr}(H)^n \tag{34}$$

For a kernel bounded by M, every row vector formed by n of the H's with different arguments cannot be larger than $M \cdot n^{1/2}$, and the determinant cannot be larger than the volume spanned by orthogonal vectors of that length: $M^n \cdot n^{n/2}$. Hence D is analytic in λ, because the Taylor series (33) converges for all λ. The eigenvalues of H are the poles of G. These are the zeros of D, so this function is a characteristic equation for the eigenvalues of H, and is called the Fredholm determinant of the operator.

If one now makes the delta function in the kernel sharp, the traces of powers of H are simply

$$\operatorname{tr}(H)^n = \sum_{x \in \operatorname{Fix} f^n} \frac{\exp \sum_{k=0}^{n-1} A(f^k(x))}{\det(1 - DF^{-n}(x))} \tag{35}$$

where the denominator arises from the integration over the delta function. One sees here that if any of the eigenvalues of a periodic orbit is marginal, the expression blows up, so hyperbolicity is a necessary assumption. In one dimension the denominator can be written out as an infinite sum

$$\frac{1}{\det(1 - DF^{-n}(x))} = \sum_{l=0}^{\infty} [DF^{-n}(x)]^l \tag{36}$$

If one thus forms the generating funtion by summing over all orders, one obtains a double sum over prime orbits and l:

$$\begin{aligned}
&\sum_{n=1}^{\infty} z^n \operatorname{tr}(H)^n \\
&\quad = \sum_{l=0}^{\infty} \sum_{p} p \left(z^{|p|} \left\{ \exp\left[\sum_{k=0}^{p-1} A(f^k(x_p)) \right] \right\} D_p^{-l} \right. \\
&\quad\quad \left. + z^{2|p|} \left\{ \exp\left[2 \sum_{k=0}^{p-1} A(f^k(x_p)) \right] \right\} D_p^{-2l} + \cdots \right) \\
&\quad = -z \frac{d}{dz} \sum_{l=0}^{\infty} \log \prod_{p} \left(1 - z^p \left\{ \exp\left[\sum_{k=0}^{p-1} A(f^k(x)) \right] \right\} D_p^{-l} \right) \\
&\quad = -z \frac{d}{dz} \log \prod_{l=0}^{\infty} \prod_{p} \left(1 - z^p \left\{ \exp\left[\sum_{k=0}^{p-1} A(f^k(x)) \right] \right\} D_p^{-l} \right) \\
&\quad = -z \frac{d}{dz} \log Z(z, A)
\end{aligned} \tag{37}$$

The notation Z for the infinite product of zeta functions with decreasing powers of the cycle derivatives is borrowed from the last section. The infinite product of Ruelle zeta functions is therefore the limit as the delta functions are made sharp of Fredholm determinants, each of which is entire analytic in z.

In the absence of proof, one how has to turn to plausibility arguments for why the limit is not singular. First, although the classical argument for the analyticity of $\det(1 - zH)$ does not work in the limit, since the bound from the maximum of the kernel diverges, each antisymmetrized trace, being sums of products of terms like Eq. (35), remains finite. Broadening the delta function is equivalent to adding noise to the deterministic dynamics,

$$(H_\sigma \phi)(x) = \left\langle \int \sum_{y = f^{-1}(x) + \xi} \delta(z - y) e^{A(z)} \phi(z)\, dz \right\rangle_\xi \tag{38}$$

so the trace of a power of H_σ is a sum over noisy periodic orbits, averaged over the noise. It is generally taken for granted that hyperbolic systems should be stable against noise, and if one supposes that this is so also in this case, the sharp delta functions will not differ markedly in effect from the broadened ones.

On the other hand, if the noise level is sufficiently small, each orbit with noise can be shadowed closely by a deterministic trajectory, with accuracy independent of the length of the orbit, so $\mathrm{tr}(H_\sigma^n)$ must change smoothly as σ goes to zero. One may therefore try to do perturbation theory. Taking for simplicity δ_σ to be a Gaussian with half-width σ, one has

$$H_\sigma(x) = \sum_{y = f^{-1}(x)} \delta(z - y) e^{A(z)} + \sum_{l=1}^{\infty} \frac{\sigma^{2l}(2l-1)!!}{(2l)!} \delta^{(2l)}(z - y) e^{A(z)} \tag{39}$$

where $\delta^{(2)}$, etc., is short-hand for the operator that extracts the second derivative. Hence the sum in Eq. (37) will only change by order σ^2 in its region of convergence (below the inverse of the lowest eigenvalue of H), and for C^∞ functions, the perturbation series in σ implied for the Fredholm determinant is convergent in that same region. This certainly suggests that the same statement holds in the entire z plane, but needless to say does not constitute proof.

As a second argument, I appeal to rigorous results by Ruelle that seem to point in the same direction.[15] His first results suppose that both the dynamics f and the weighting function $\varphi = e^A$ are real analytic, and show that if so, the zeta functions extend to meromorphic functions in z, with isolated leading zero, which can be written as quotients of two entire analytic functions. It is not clear how important analyticity is; periodic orbits and their eigenvalues are invariants of even C^1 conjugacy, while it would be prudent to assume that analyticity is not.[2]

To arrive there, Ruelle considers operators that act on k-forms ω as

$$(L_k\omega)(z) = \sum_{y=f^{-1}(z)} \varphi(y)\{[A^k(f'(y))^{-1}]\,\omega(y)\} \tag{40}$$

where $A^k(f'(y))^{-1}$ is the totally antisymmetrized product of k copies of the inverse derivative matrix $f'(y)^{-1}$. The operators L_k turn out to be of a type called nuclear, to which Fredholm's theory has been generalized, and which have determinants that are entire analytic funtions of z. If one remembers that

$$\det\{1-[f'(y)]^{-1}\} = \sum_k (-1)^k A^k(f'(y))^{-1}$$

it follows that the zeta function of Ruelle can be written as the product

$$\zeta^{-1}(z, A) = \frac{\prod_{k\,\mathrm{odd}} \det(1-zL_k)}{\prod_{k\,\mathrm{even}} \det(1-zL_k)} \tag{41}$$

In one dimension there are only 0-forms and 1-forms, so the formula simplifies considerably to

$$\zeta^{-1}(z, A) = \frac{\det(1-zL_0)}{\det(1-zL_1)} \tag{42}$$

If one is willing to disregard that the operators L_0 and L_1 act on different spaces, one would say that in one dimension it looks as if $\det(1-zL_1)$ is just like $\det(1-zL_0)$, but with a different weight, i.e., one power of the derivative, with sign, less in the weighting function. In all one would like to identify $\det(1-zL_0)$ with $Z(z, A)$, the infinite product of zeta functions with decreasing powers of the derivative. This is not then a formal identity, because each factor in $Z(z, A)$ has a nice power series expansion, so it can be conveniently calculated and its poles and zero determined through Padé approximants. The entire function can then be assembled from the products of its factors, where the tailing factors go rapidly to 1 for any reasonable z.

[2] Recent results show, indeed, if I understand them properly, that C^∞ functions are sufficient.[16]

4. PRUNING AMONG THE COMPLEXES

In most dynamical systems coded by symbolic dynamics, there are transitions that are not realized by the motion, or sequences that are pruned from the set of all words in the symbols.[17] I here consider a (simply) pruned symbolic dynamics, and show that the present definition of complexes makes sense and is of some use also in this case. Consider the golden mean pruning (i.e., 11 not allowed in a binary sequence) (see Fig. 6). The $N = 1$ diagram has the forbidden sequence on one of its links (the link from 1 to 1 that has been pruned), and is the smallest diagram that correctly produces all words allowed. Call it the basic diagram of the golden mean pruning. If the order of a complex is greater than or equal to the order of the basic diagram, the complex is either entirely pruned away or remains unscathed, depending on whether its graph passes the forbidden link on the basic diagram. If the order is smaller, some radicals, all radicals, or no radicals in a complex may be pruned away. For golden mean pruning, 0, 1, and 01 are the only complexes with order lower than 1, and

$$\begin{aligned} \langle 1 \rangle &\quad \text{is entirely pruned} \\ \langle 0 \rangle &\quad \text{is not pruned at all} \\ \langle 01 \rangle = 01 - 0 * 1 &\quad \text{is pruned to } 01 \end{aligned} \tag{43}$$

It is the partly pruned complexes that determine the equation for the topological entropy

$$0 = \prod_p (1 - z^p) \tag{44}$$

since the complexes that are entirely pruned or not pruned at all do not contribute, by construction. As a consequence, it follows that if one evaluates

$$\prod_p (1 - z^p t_p) = \sum_{k=0}^{\infty} z^k c_k \tag{45}$$

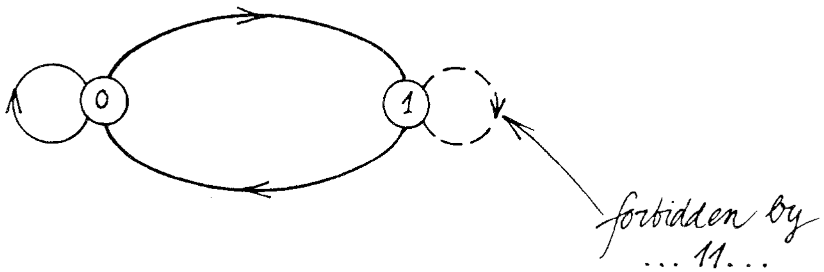

Fig. 6. The basic diagram of golden mean pruning.

by truncation in k, all terms with k larger than the order of the equation for the topological entropy correspond to unpruned complexes, and should be small. Convergence should therefore be smooth, and by our earlier considerations exponential, beyond the order of the entropy polynomial, if it is finite. In general one may expect there to be many pruning rules: each rule then prunes the complexes in turn, so one can consider them acting one at time, the shorter ones that give the gross structure first. In a pruned symbolic dynamics on two symbols, the degree of the equation for the topological entropy is less than 2^{n+1}, if n is the order of the basic diagram. Although this bound is not usually saturated, it indicates that when there are many pruning rules of large order, exponential convergence of the zeta function is lost.

5. DISCUSSION

Hyperbolic systems in any number of dimensions have Markov partitions, i.e., well-defined symbolic dynamics.[18] Therefore, one should be able to extend the analysis in Sections 2–4, introduce complexes and radicals, and eventually define curvatures. However, if one wants to evaluate a quantity that needs one eigenvalue of the linearized map around an orbit, for instance, the largest if one calculates the leading Lyapunov exponent, the estimates of sizes of complexes as in Section 2 can sometimes be quite wrong in the higher-dimensional case. This is so because in a pair of radicals as in Fig. 4, even though the two linearized maps are close,

$$T_{ab} - T_a \cdot T_b \sim 0 \tag{46}$$

the eigenvalues are extracted orbit by orbit in the "counterterm," and $\lambda^1(T_{ab})$ does not have to be close to $\lambda^1(T_a) \cdot \lambda^1(T_b)$. Effectively, such mismatches should act as additional pruning rules and dominate the convergence rate.[3]

In nonhyperbolic systems the basis of the analysis in Section 2 falls apart, as one cannot assume that the difference between two radicals is small. This is illustrated by the simple example of the Ulam map that maps the unit interval to itself by $x \mapsto 4x(1-x)$. It is standard knowledge that this map is conjugate to a linear tent map $x \mapsto 1 - 2\,|x-(1/2)|$, smoothly everywhere except at $x=0$ and $x=1$. The derivative at the fix-point at

[3] At least such was the case in a calculation of the Hausdorff dimension of a two-dimensional repeller, where the least expanding eigenvalue is the relevant one.[19] In a map with two expanding eigenvalues, they can either be both real or complex conjugate. If the latter happens to one member in a pair of radicals, effectively the faster expansion rate is mixed with the slower, and the offset between the two terms will be large.

$x=0$ is 4, but around all other cycles it agrees with the linear map, i.e., 2^p in absolute value, if p is the length of the cycle in binary digits. All complexes that have "0" as a crossing-point are therefore never small: close passages to the critical point at $x=1/2$ distort the derivatives, and the arguments of Section 2 do not apply. The zeta functions can be calculated analytically to be[20]

$$\zeta^{-1}(z,\tau)=(1-z4^{\tau})\prod_{p\neq 0}(1-z^p\,|D_p|^{\tau})$$
$$=\frac{(1-z4^{\tau})(1-z2^{\tau+1})}{1-z2^{\tau}}$$

and

$$\zeta_{-1}^{-1}(z,\tau-1)=(1-z4^{\tau-1})\prod_{p\neq 0}[1-z^p\,|D_p|^{\tau}\,(D_p)^{-1}]$$
$$=\frac{1-z4^{\tau-1}}{1-z2^{\tau-1}} \tag{47}$$

and, although rational, they do not satisfy the same relation between the zeros and poles as the hyperbolic zeta functions. Depending on the value of τ, the leading zero in ζ^{-1} comes either from the fix-point to the left or the product over the other points. One can consider the fix-point at $x=0$ as a special "phase," as it is the image of the critical point and there the measure has a square-root singularity, while the other periodic points form a normal hyperbolic phase that spans the rest of the interval. In addition, the preimages of the critical point can be considered as a nonhyperbolic phase, since for a finite number of iterations, the exponential divergence of nearby trajectories is killed by the quadratic contraction at the critical point. This phase is not seen in the periodic orbits, since contraction around a periodic orbit would imply that the orbit is attractive, and then the map could not be chaotic, so not all averages are calculable from zeta functions in the nonhyperbolic case.[7]

The normal hyperbolic phase has the peculiar feature that *infinite* long strings of 0's are forbidden (since that is the symbolic address of the fix-point at $x=0$), or in other words: there is a pruning rule of infinite length, which is responsible for the denominator in the zeta function (47). The convergence of average quantities in the normal phase in this example is therefore seen to be dominated not by curvatures (of which there are none, since all its periodic orbits have the same instability exponents), but by prunings in the remaining hyperbolic phase, when the special fix-point at $x=0$ is separated out. One would expect that both features persist in less trivial nonhyperbolic maps: there would be not one, but many points that compress points so that hyperbolicity is lost, and an effort to determine which orbits are pruned, or which are not, and if any "critical" points iterate onto periodic orbits or nearly so, brings in all the complexity of the description of nonhyperbolic systems. In addition, there could always be stable or almost stable periodic orbits of very long length, so high coefficients in the power series expansions of ζ^{-1} are sensitive to minute changes in the map brought on by very small changes in the parameters.

In conclusion, I conjecture that converge is dominated by curvatures, in hyperbolic systems, when the quantity evaluated along a periodic orbit is multiplicative over the points on the orbit. This includes derivatives in the one-dimensional case, and products over all the expanding or all the contracting eigenvalues of the derivative map in the higher-dimensional case. Then convergence is exponential, limited by a pole, and the location of the pole can be calculated. In the hyperbolic phase of nonhyperbolic systems, or in the evaluation of nonmultiplicative functions in hyperbolic systems, I conjecture that convergence is dominated by real or effective pruning rules of arbitrary length. The concept of curvature would therefore seem to be less useful in that case.

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